Sem V

MJC 08

08.11.2025

Ring Homomorphism

Definition

Let R and R' be two ring.

Let $f: R \to R'$. Then f is called a homomorphism if

$$f(a+b) = f(a) + f(b) \quad \forall a, b \in R$$

$$\text{and } f(ab) = f(a)\, f(b) \quad \forall a, b \in R.$$

If f is onto then we say that R' is the homomorphic image of R.

It is denoted by $R \simeq R'$.

Theorem Q→ If f is a homomorphism of a ring R into another ring R' then prove that

(i) $f(0) = 0'$ where 0 and $0'$ are zero elements of R and R' respectively.

(ii) $f(-a) = -f(a) \quad \forall a \in R$

Proof (i) Let $a \in R \Rightarrow f(a) \in R'$

Since $0'$ is identity element of ring R'

$\Rightarrow f(a) + 0' = f(a)$ —— (1)

Also, 0 is additive identity element of R

$\Rightarrow a + 0 = a \quad \forall\, a \in R$ —— (2)

So (1) becomes

$$f(a) + 0' = f(a)$$
$$= f(a+0) \quad [\text{using } (2)]$$
$$= f(a) + f(0) \quad [\text{By definition of homomorphism}]$$

$\therefore f(a) + 0' = f(a) + f(0)$ —— (3)

$\because (R', +)$ is a group, by definition of ring.

$\Rightarrow f(a) + 0' = f(a)$ as $f(a) \in R'$, $0'$ is additive identity element of R'

~~Putting this value in (3)~~

$\Rightarrow f(a) + f(0) = f(a) + 0'$ using left cancellation law

we have $f(0) = 0'$ Proved

(ii) Let $a \in R \Rightarrow -a \in R$

$\therefore a + (-a) = 0$, (by definition) —— (4)

$\because f(0) = 0' \Rightarrow f[a + (-a)] = 0'$ [using (4)]

$\Rightarrow f(a) + f(-a) = 0'$

$\Rightarrow f(-a)$ is additive inverse of $f(a) \Rightarrow f(-a) = -f(a)$

Proved